Forced harmonic vibration of a Duffing oscillator

In 1918, Duffing introduced a nonlinear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical problems (see [2] for a review). Since then this equation has become, together with Van der Pol's equation, one of the commonest examples in nonlinear oscillation texts and research articles.

As formalized by Duffing, the stiffness is generally a function of position. This means that the force applied to the spring F_s, and the resulting **displacement** y have a nonlinear relationship. If the system is symmetric, i.e., the stiffness characteristic is the same when the spring is in compression or in tension, then the restoring force can be approximated as a series in y in which the exponents of y are odd integers. If this series is truncated after the first two terms then the force–deflection relationship is given in non-dimensional form by

$$\tilde{F}_s = \tilde{y} + \gamma \tilde{y}^3, \tag{1}$$

, and illustrated in fig. 1. A positive (negative) **cubic stiffness parameter** γ corresponds to a hardening (softening) spring.

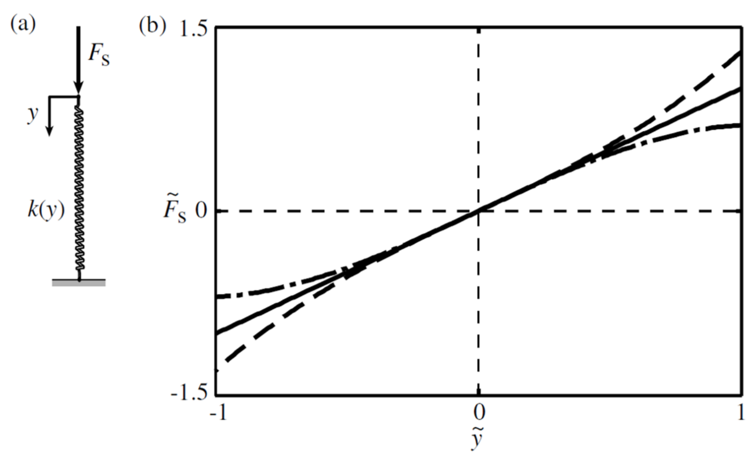

Figure 1: Characteristics of a nonlinear hardening and softening spring described by (1). (a) A nonlinear spring in which the stiffness is a function of the displacement; (b) Force deflection characteristics for a linear spring (solid line), hardening spring with $\gamma = 0.3$ (dashed line) and softening spring with $\gamma = -0.3$ (dashed-dotted line). Figure reproduced from [2]

Therefore, the nondimensional Duffing equation (dropping the tildes) with damping and harmonic external forcing studied in this chapter has the form

$$\begin{aligned} \ddot{y} + 2\zeta\dot{y} &= F\cos(\Omega t) - F_s \Leftrightarrow \\ \ddot{y} + 2\zeta\dot{y} + y + \gamma y^3 &= F\cos(\Omega t) \end{aligned} \tag{2}$$

where t**,** ζ**,** F **and** Ω **are the time, damping ratio, excitation amplitude and excitation frequency**, respectively. Note that with $\gamma = 0$, (2) reduces to the forced linear oscillator. The goal of this exercise is to understand the influence of nonlinearity and compare the behaviour of the forced nonlinear system with that of the forced linear system.

To facilitate the nonlinear analysis, a small parameter $\epsilon \ll 1$, is introduced as an asymptotic ordering parameter and the damping and nonlinear terms are written, respectively, as $\zeta = \epsilon\overline{\zeta}$ and $\gamma = \epsilon\overline{\gamma}$, where $\overline{\zeta}$ and $\overline{\gamma}$ are $O(1)$ quantities. A weak or soft forcing $F = \epsilon\overline{F}$, where $\overline{F}$ is $O(1)$ is also assumed. With these assumptions, (2) can be rewritten as

$$\ddot{y} + y + \epsilon(2\overline{\zeta}\dot{y} + \overline{\gamma}y^3) = \epsilon\overline{F}\cos(\Omega t) \tag{3}$$

Next, the system response during the resonance excitation, $\Omega \approx 1$, is considered. The proximity of the excitation frequency to the system natural frequency is expressed as

$$\Omega = 1 + \epsilon\sigma \tag{4}$$

where σ **is called the detuning parameter**, which is a measure of how close the excitation frequency is to the natural frequency. Eventually, (3) is rewritten

$$\ddot{y} + y + \epsilon(2\overline{\zeta}\dot{y} + \overline{\gamma}y^3) = \epsilon\overline{F}\cos((1 + \epsilon\sigma)t) \tag{5}$$

Analytical approximations of the solution of (5) can be constructed by using the method of multiple scales. Let

$$y(t;\epsilon) = y_o(T_0, T_1) + \epsilon y_1(T_0, T_1) + O(\epsilon^2) \tag{6}$$

where the fast *timescale* T_0 and the slow timecale T_1 are given by

$$T_0 = t, \quad T_1 = \epsilon t. \tag{7}$$

With the introduction of the timescales, the time derivative with respect to time t is transformed as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} \tag{8}$$

Q1) Show that the $O(1)$ solution reads

$$y_0(T_0, T_1) = A(T_1)e^{iT_0} + A^*(T_1)e^{-iT_0} \tag{9}$$

where $A(T1)$ is a complex valued amplitude function, undetermined for now, and * indicates a complex conjugate of that quantity.

Q2) Show that the $O(\epsilon)$ equation is

$$\frac{\partial^2 y_1}{\partial T_0^2} + y_1 = -2\frac{\partial^2 y_0}{\partial T_0 \partial T_1} - 2\overline{\zeta}\frac{\partial y_0}{\partial T_0} - \overline{\gamma} y_0^3 + \overline{F}\cos(T_0 + \sigma T_1) \tag{10}$$

Q3) Deduce that the amplitude $A(T_1)$ must satisfy

$$-i(2A' + 2\overline{\zeta}A) - 3\overline{\gamma}A^2 A^* + \frac{\overline{F}}{2}e^{i\sigma T_1} = 0 \tag{11}$$

Q4) Introducing now the polar form of the complex amplitude

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)} \tag{12}$$

where the amplitude $a(T_1)$ and the angle $\beta(T_1)$ are real-valued quantities, show that (11) can be rewritten as

$$\begin{aligned} a' &= -\overline{\zeta}a - \frac{\overline{F}}{2}\sin\phi \\ a\phi' &= -\left(\sigma a - \frac{3}{8}\overline{\gamma}a^3 + \frac{\overline{F}}{2}\cos\phi\right) \end{aligned} \tag{13}$$

where $\phi(T_1) = -(\sigma T_1 - \beta)$

Q5) The equilibrium (or "fixed") points of (13) correspond to solutions with constant amplitude and phase. Show that the equilibrium amplitude solves

$$\bar{F}^2 = 4a^2\left(\overline{\zeta}^2 + \left(\sigma - \frac{3}{8}\overline{\gamma}a^2\right)^2\right) \tag{14}$$

Q6) Solve (14) numerically (be aware that several solutions may exist) by choosing $\overline{F} = 0.3$, $\overline{\zeta} = 0.1$ and $\epsilon = 0.2$, for $\overline{\gamma} \in [-3, -1, 0, 1, 5]$ and $\Omega \in [0.5; 2]$. Plot the response-to-forcing gain (or "magnification factor")

$$M = \frac{a}{|\overline{F}|} \tag{15}$$

as a function of Ω. What do you observe ? What are the main difference(s) with respect to the linear oscillator (corresponding to $\overline{\gamma} = 0$) ?

Q7) Read the recent research paper [1] and comment on the analogy between a sloshing fluid flow and the forced Duffing oscillator.

References

[1] Bongarzone, A., Guido, M., and Gallaire, F. An amplitude equation modelling the double-crest swirling in orbital-shaken cylindrical containers. *Journal of Fluid Mechanics 943* (2022), A28.

[2] Brennan, M. J., and Kovacic, I. *Examples of Physical Systems Described by the Duffing Equation.* John Wiley & Sons, Ltd, 2011, ch. 2, pp. 25–53.